THEORY OF COMPUTATION

UNIT-I

SETS, RELATIONS AND FUNCTIONS

SETS

A set is a well-defined collection of objects. We use the capital letters A, B, C ... for denoting sets. The small letters a, b, c ..... are used to denote the elements of any set. When 'a' is an element of set A, we write $a \in A$. When 'a' is not an element of A, we write $a \notin A$.

Example: $A = \{1, 2, 3, 4, 5\}$

Here A is the set and 1, 2, 3, 4, 5 are the elements of the set or a member of a set. The elements that are written in the set are in any order and it cannot be repeated. Also we can write $1 \in A$, $2 \in A$, $3 \in A$ etc.

The number of elements in the finite set is represented as the Cardinal Number (or Cardinality) of a set or Order of a set.

In the above example the Cardinal number of the set is 5.

Some Commonly used sets are as follows:

N: Set of all natural ~~aa~~ numbers.

Z: Set of all integers

Q: Set of all rational numbers

R: Set of all real numbers.

Types of Set

1) Empty sets

A set which doesnot Contain any element is Called an empty set or void set or null set.
It is denoted by { } or ϕ

2) Singleton Set

A set which Contains a single element is Called a singleton set.

eg: A = {a}

3) Finite Set

A set which Contains definite number of elements is called a finite set.

eg: A = {2, 4, 6, 8}

4) <u>Infinite Set</u>

A set which is not finite is called infinite set.

eg: A set of all natural numbers.

$A = \{1, 2, 3, 4 \ldots\}$

5) <u>Equivalent Set</u>

If the numbers of elements are same for two different sets, then they are called Equivalent sets. The order of sets doesnot matter here.

It is represented as:

$n(A) = n(B)$, where A & B are two different sets with same number of elements

eg: $A = \{1, 2, 3, 4\}$, $B = \{$Red, Blue, White, Black$\}$

Here A & B are equivalent sets.

6) <u>Equal Sets</u>

The two sets A and B are said to be equal if they have exactly the same elements, order of the elements do not matter.

eg: $A = \{1, 2, 3, 4\}$, $B = \{4, 3, 2, 1\}$

$A = B$

7) Disjoint Sets

The two sets A and B are said to be disjoint if the set does not contain any common element.

eg: $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7, 8\}$

Here A & B are Disjoint.

Representations of Set

1) Roster Form : all the elements of a set are listed

eg: Set of all natural numbers less than 5

$A = \{1, 2, 3, 4\}$

2) Set-Builder form : It describes the properties of the elements of the set.

The general form is $A = \{x : \text{property}\}$

The set $\{1, 2, 3, 4\}$ can be represented as

$A = \{n \mid n$ is the set of all natural numbers less than $5\}$

or

$A = \{n \mid n \in N, n < 5\}$

Subsets

A Set A is Said to be a Subset of B if every element of A is also an element of B denoted by $A \subseteq B$. Even the null set is considered to be the subset of another set.

eg: $A = \{1, 2, 3\}$

Then, $\{2, 3\} \subseteq A$, $\{1\} \subseteq A$, $\{1, 2, 3\} \subseteq A$

Similary other Subsets are: $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{\}$

If A is not a subset of B, then it is denoted as $A \not\subseteq B$.

Power Set

For a set A, a collection or family of all subsets of A is called the power set of A. The power set of A is denoted by P(A).

eg: $A = \{1, 2, 3\}$

$$P(A) = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}, \{\}\}$$

The number of elements/ subsets in the power set is given by 2^N, where N is the number of elements in the given set.

Proper Subset

If $A \subseteq B$ and $A \neq B$, then A is called the proper subset of B and can be written as $A \subset B$

eg: $A = \{2, 5\}$, $B = \{2, 5\}$ → Not proper subset

$A = \{2, 5\}$, $B = \{2, 5, 3\}$ → Proper Subset.

Superset

A set 'A' is a superset of another set B if all elements of set B are elements of the set A. The superset relationship is denoted as $A \supset B$.

eg: $A = \{1, 2, 3\}$, $B = \{1, 2\}$

Here, $A \supset B$.

Universal Set

A set which contain all the sets relevant to a certain contain condition is called universal set. It is the set of all possible values.

eg: $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$

$U = \{1, 2, 3, 4, 5\}$

Operations on Sets

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$, called the Union of A and B

$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$, called the Intersection of A and B

$A - B = \{x \mid x \in A \text{ and } x \notin B\}$, called the Complement of B in A.

A^c (or A') denotes $U - A$, where U is the Universal Set, the set of all elements under consideration

Example: Let $A = \{1, 3, 4, 5\}$, $B = \{2, 4, 6\}$, $U = \{1, 2, 3, 4, 5, 6\}$

Then,

$A \cup B = \{1, 2, 3, 4, 5, 6\}$

$A \cap B = \{4\}$

$A - B = \{1, 3, 5\}$

$A^c = \{2, 6\}$

Let A and B be two sets. Then $A \times B$ is defined as $\{(a,b) \mid a \in A \text{ and } b \in B\}$. (a,b) is called an ordered pair and is different from (b,a).

eg: $A = \{1, 2\}$, $B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$B \times A = \{(3,1), (3,2), (4,1), (4,2)\}$

Definition :

Let S be a Set. A Collection $(A_1, A_2, \ldots, A_n)$ of Subsets of S is called a Partition if $A_i \cap A_j = \phi$ $(i \neq j)$ and $S = \bigcup_{i=1}^{n} A_i$ (i.e, $A_1 \cup A_2 \cup \ldots \cup A_n$).

For example, if $S = \{1, 2, 3, 4 \ldots 10\}$, then $\{\{1, 3, 5, 7, 9\}, \{2, 4, 6, 10\}\}$ is a partition of S.

Sets with one Binary Operation

A binary operation $*$ on a set S is a rule which assigns, to every ordered pair (a, b) of elements from S, a unique element denoted by ~~$a*b$~~. $a*b$

Addition, for example, is a binary operation on the Set Z of all integers.

The Five postulates on binary operations are :

1) Closure : If a and b are in S, then $a*b$ is in S.

eg: If $S = \{1, 2, 3, 4 \ldots\}$

Then, if, $a = 3$, $b = 4$

then, $a+b = 3+4 = 7 \in S$

(Here $*$ represents '+' operation)

2) <u>Associativity</u>: If a, b, c are in S, then

$(a*b)*c = a*(b*c)$

3) <u>Identity element</u>: There exist a unique element (called the identity element) e in S such that for any element x in S

$x*e = e*x = x$

eg: If $x = q$ and $*$ represents '$\times$' (multiplication) then,

$x \times 1 = 1 \times x = x$

$\therefore$ The identity element 'e' for multiplication is 1.

4) <u>Inverse</u>: For every element x in S there exists a unique element x' in S such that $x * x' = x' * x = e$. The element x' is called the inverse of x with respect to (w.r.t) $*$.

eg. If $x = 3$, $x' = 1/3$

then,

$3 \times 1/3 = \frac{1}{3} \times 3 = e$ (value of e being 1)

Here $*$ represents multiplication

5) Commutativity: If $a, b \in S$, then $a * b = b * a$

It may be noted that a binary operation may satisfy none of the above five postulates For example, let $S = \{1, 2, 3, 4 \ldots\}$ and let the binary operation be substraction ie, $a * b = a - b$. The Closure postulate is not satisfied since $2 - 3 = -1 \notin S$.

Definitions

i) A set S with binary operation * is called a semigroup if the postulates 1 and 2 are satisfied.

ii) A set S with a binary operation * is called a monoid if the postulates 1-3 are satisfied.

iii) A set S with * is called group if the postulates 1-4 are satisfied.

iv) A group is called Commutative or Abelian group if the postulate 5 is satisfied

2.1.3 SETS WITH TWO BINARY OPERATIONS

Sometimes we come across sets with two binary operations defined on them (for example, in the case of numbers we have addition and multiplication). Let S be a set with two binary operations $*$ and o. We give below 11 postulates in the following way:

(i) Postulates 1–5 refer to $*$ postulates.

(ii) Postulates 6, 7, 8, 10 are simply the postulates 1, 2, 3, 5 for the binary operation o.

(iii) *Postulate* 9: If S under $*$ satisfies the postulates 1–5 then for every x in S, with $x \neq e$, there exists a unique element x' in S such that $x' \circ x = x \circ x' = e'$, where e' is the identity element corresponding to o.

(iv) *Postulate* 11: *Distributivity*. For a, b, c, in S

$$a \circ (b * c) = (a \circ b) * (a \circ c)$$

A set with one or more binary operations is called an algebraic system. For example, groups, monoids, semigroups are algebraic systems with one binary operation.

We now define some algebraic systems with two binary operations.

Definitions (i) A set with two binary operations $*$ and o is called a *ring* if (a) it is an abelian group w.r.t. $*$, and (b) o satisfies the closure, associativity and distributivity postulates (i.e. postulates 6, 7 and 11).

(ii) A ring is called a commutative ring if the commutativity postulate is satisfied for o.

(iii) A commutative ring with unity is a commutative ring that satisfies the identity postulate (i.e. postulate 8) for o.

(iv) A *field* is a set with two binary operations $*$ and o if it satisfies the postulates 1–11.