Determinants

The determinant is a scalar value that can be computed from the elements of a square matrix, or we can say that every square matrix is associated with a number and this number is known as Determinant.

The determinant of a matrix A is denoted by $|A|$, det A or Δ

Determinant of a Square matrix of order 1 (1x1)

Let $A = [a]$ then the determinant of A is defined as

$$|A| = |a| = a$$

eg: $A = [3]$

Then, $|A| = |3| = \underline{\underline{3}}$

eg: $A = [-3]$

$|A| = |-3| = \underline{\underline{-3}}$

Determinant of a square matrix of order 2 (2×2)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a square matrix

of order 2, then the determinant of A is defined as

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

eg: $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 2\times 1 - 3\times 5 = 2-15 = \underline{\underline{-13}}$$

eg: $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} = 1\times 4 - (-2\times 3) = 4+6 = \underline{\underline{10}}$

Determinant of the square matrix of order 3 (3×3)

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Determinant of A is defined as

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-1)^{1+1} a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & 33 \end{vmatrix} + (-1)^{1+2} a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + (-1)^{1+3} a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}\left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12}\left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13}\left(a_{21}a_{32} - a_{22}a_{31}\right)$$

is known as the expansion of the determinant along the first Row. Since the matrix A contains 3 rows and 3 columns there are 6 ways for expanding the determinant which gives the same value

While expanding a 3^{rd} order determinant by any row or column, the signs to be taken before each of the three terms are given in the diagram

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

* Evaluate the determinant of the matrix

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}$$

Sol: Expanding along the first row, we have,

$$|A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$$

$$= (-1)^{1+1} \times 3 \begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} + (-1)^{1+2} \times -1 \begin{vmatrix} 0 & -1 \\ 3 & 0 \end{vmatrix} + (-1)^{1+3} \times -2 \begin{vmatrix} 0 & 0 \\ 3 & -5 \end{vmatrix}$$

$$= 3 \left[0 \times 0 - (-1 \times -5) + -1 \times -1 \left[0 \times 0 - (-1 \times 3) \right] \right.$$

$$\left. + -2 \left[0 \times -5 - 0 \times 3 \right] \right]$$

$$= 3\left[0-5\right] + 1\left[0+3\right] - 2\left[0-0\right]$$

$$= -15 + 3 - 0 = \underline{\underline{-12}}$$

$$\therefore |A| = \underline{\underline{-12}}$$

Similarly we can find the determinet by expanding along the 2nd row.

$$|A| = (-1)^{2+1} \times 0 \begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + (-1)^{2+2} \times 0 \begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} + (-1)^{2+3} \times -1 \begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix}$$

$$= -1 \times 0 \left(-1 \times 0 - (-2 \times -5)\right) + 1 \times 0 \left[3 \times 0 - (-2 \times 3)\right]$$

$$+ -1 \times -1 \left[3 \times -5 - 3 \times -1\right]$$

$$= 0[0-10] + 0[0+6] + 1[-15+3]$$

$$= 0 + 0 - 12 = \underline{\underline{-12}}$$

$$\therefore |A| = \underline{\underline{-12}}$$

* $A = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$. Find $|A|$.

Sol: $|A| = \begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

$$= 3 \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - -4 \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3[1 - -6] + 4[1 - -4] + 5[3 - 2]$$

$= 3\left[1+6\right] + 4\left[1+4\right] + 5 \times 1$

$= 3 \times 7 + 4 \times 5 + 5$

$= 21 + 20 + 5$

$= \underline{\underline{46}}$

* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Find $|A|$

Sol: $|A| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

$= \cos\theta \times \cos\theta - (-\sin\theta \times \sin\theta)$

$= \cos^2\theta + \sin^2\theta$

$= \underline{\underline{1}}$

Note: If A is any square matrix of order n then

$|KA| = K^n |A|$

* If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ then show that $|3A| = 3^3 |A|$

Sol: Given, $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$

$$3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

$$|3A| = \begin{vmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{vmatrix}$$

$$= 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix} + 0\begin{vmatrix} 3 & 3 \\ 0 & 6 \end{vmatrix} + 12\begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix}$$

(expanded on 3^{rd} row)

$$= 0 + 0 + 12\,[3\times 3 - 0]$$

$$= 12 \times 9 = \underline{\underline{108}} \quad \text{--- (1)}$$

Now, $|A| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$

$$= 0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 0 + 0 + 4\,[1 - 0]$$

$$= \underline{\underline{4}}$$

Now,

$$3^3|A| = 27 \times 4 = \underline{108} \quad \text{—} \quad (2)$$

from (1) and (2)

It is clear that $|3A| = 3^3|A|$

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* Find the value of x if $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

Sol: Now,

$$\begin{vmatrix} 2 & x \\ 5 & 1 \end{vmatrix} = 2 \times 1 - 5 \times 4$$

$$= 2 - 20 = \underline{-18}$$

$$\begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix} = 2x \times x - 6 \times 4$$

$$= 2x^2 - 24$$

By given condition,

$$-18 = 2x^2 - 24$$

$$\Rightarrow 2x^2 = -18 + 24$$

$$\Rightarrow 2x^2 = 6$$

$$\Rightarrow x^2 = 6/2$$

$$\Rightarrow x^2 = 3 \Rightarrow \underline{x = \sqrt{3}}$$